Vector Partitions and Combinatorial Identities

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In this note we show how certain relations between vector partition functions can be deduced from certain identities. A relation connecting vector partitions having odd components and those having distinct parts will be proved. A combinatorial proof of Jacobi's Identity similar to Franklin's proof of Euler identity is suggested. The last section includes numerical values of $P_r(n, m)$ and $q_r(n, m)$. These results suggest the unique maxima property of $P_r(n, m)$ for fixed n, m and r varying.

In the Jacobi Identity

$$(1.1) \qquad \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}t)(1 + q^{2n-1}t^{-1}) = \sum_{-\infty}^{+\infty} q^{n^2}t^n$$

make the substitution $q^2 = xy$, $t^2 = x/y$ and change x to $-x$, y to $-y$ to obtain

$$(1.2) \quad \prod_{n=1}^{\infty} (1 - x^n y^n)(1 - x^n y^{n-1})(1 - x^{n-1}y^n) = \sum_{-\infty}^{+\infty} (-1)^n x^{n(n+1)/2} y^{n(n-1)/2}.$$

This when interpreted combinatorially yields the following

Theorem I. *The excess of the number of partitions of (n, m) into even number of distinct parts of the type $(a, a - 1)$, $(b - 1, b)$, (c, c) over those into odd number of such parts is $(-1)^r$ or 0 according as (n, m) is of the type $(r(r + 1)/2, r(r - 1)/2)$ or not.*

Let $\alpha(n, m)$ denote the number of partitions of (n, m) into distinct parts $(a, a - 1)$, $(b - 1, b)$ so that we have the generating function

$$\sum_{n,m=0}^{\infty} \alpha(n, m)x^n y^m = \prod_{n=1}^{\infty} (1 + x^n y^{n-1})(1 + x^{n-1}y^n).$$

In 1.1 making the substitution $q^2 = xy$, $t^2 = x/y$ we obtain

$$\prod_{n=1}^{\infty} (1 + x^n y^{n-1})(1 + x^{n-1}y^n) = \left\{\prod_{n=1}^{\infty} (1 - x^n y^n)\right\}^{-1} \left\{\sum_{-\infty}^{+\infty} x^{n(n+1)/2} y^{n(n-1)/2}\right\}$$
$$= \left\{\sum_{n=1}^{\infty} p(n)x^n y^n\right\}\left\{\sum_{-\infty}^{+\infty} x^{n(n+1)/2} y^{n(n-1)/2}\right\}.$$

Equating coefficients Carlitz [2] obtained

$$\alpha(n, m) = p(n - \tfrac{1}{2}(n - m)(n - m + 1)).$$

Conversely if one can prove this result combinatorially it yields a proof of Jacobi's Identity, such a proof has been obtained by Wright in a forthcoming paper by setting up a 1-1 correspondence between the two types of partitions.

This is done by placing a triangular array of $(n - m)(n - m + 1)/2$ dots on the graph of each partition of $n - \frac{1}{2}(n - m)(n - m + 1)$, the columns under the diagonal and rows on the right side determine uniquely parts $(a, a - 1)$, $(b - 1, b)$ of (n, m).

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If one can prove Theorem I by combinatorial arguments similar to Franklin's proof of Euler identity

$$\prod_{r=1}^{\infty}(1-x^r)=\sum_{-\infty}^{+\infty}(-1)^{\lambda}x^{\lambda(3\lambda+1)/2}, \tag{1.3}$$

it will yield a combinatorial proof of Jacobi's Identity. The method of proof will depend on setting up a 1-1 correspondence between the partitions into even number of distinct parts and odd number of distinct parts of the type $(a, a-1)$, $(b-1, b)$, (c, c); such a correspondence has to be 1-1 both ways. By means of simple operations one can change the parity of the number of parts except in the case when (n, m) is of the type $(r(r+1)/2, r(r-1)/2)$, the parity of whose partition $(r, r-1), (r-1, r-2), \cdots, (2,1), (1,0)$ cannot be changed and thus the excess in this case is $(-1)^r$ and 0 in other cases.

Gordon [1] has generalized Jacobi's Identity such that there are five products on the left side, i.e.,

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}t)(1-q^{2n-1}t^{-1})(1-q^{4n-4}t^2)(1-q^{4n-4}t^{-2})\\ =\sum_{-\infty}^{+\infty}q^{3n^2-2n}(t^{3n}+t^{-3n}-t^{3n-2}-t^{-3n+2}).\end{aligned} \tag{1.4}$$

Again put $q^2 = xy$, $t^2 = x/y$ to obtain

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^ny^n)(1-x^ny^{n-1})(1-x^{2n-1}y^{2n-3})(1-x^{2n-3}y^{2n-1})(1-x^{n-1}y^n)\\ =\sum_{n=-\infty}^{+\infty}\{x^{(3n^2+n)/2}y^{(3n^2-5n)/2}+x^{(3n^2-5n)/2}y^{(3n^2+n)/2}-x^{(3n^2+n-2)/2}y^{(3n^2-5n+2)/2}\\ -x^{(3n^2-5n+2)/2}y^{(3n^2+n-2)/2}\}.\end{aligned} \tag{1.5}$$

Let $C(c, m)$ denote number of partitions of (n, m) into vectors of the type (a, a), $(b, b-1)$, $(c-1, c)$, $(2d-1, 2d-3)$, $(2e-3, 2e-1)$; thus the generating function is given by

$$\begin{aligned}\prod_{n=1}\{(1-x^ny^n)(1-x^ny^{n-1})(1-x^{n-1}y^n)(1-x^{2n-1}y^{2n-3})(1-x^{2n-3}y^{2n-1})\}^{-1}\\ =\sum_{n,m=0}^{\infty}C(n,m)x^ny^m.\end{aligned} \tag{1.6}$$

Thus (1.6) yields the recurrence relation

$$\begin{aligned}\sum C\left(n-\frac{3r^2+r}{2},\ m-\frac{3r^2-5r}{2}\right)+\sum C\left(n-\frac{3r^2-5r}{2}, m-\frac{3r^2+r}{2}\right)\\ -\sum C\left(n-\frac{3r^2+r-2}{2},\ m-\frac{(3r^2+5r+2)}{2}\right)\\ -\sum C\left(n-\frac{(3r^2-5r+2)}{2}, m-\frac{(3r^2-r-2)}{2}\right)=0.\end{aligned} \tag{1.7}$$

Change x to $-x$, y to $-y$ in (1.5) to obtain

$$\prod_{n=1}^{\infty}(1-x^ny^n)(1+x^ny^{n-1})(1+x^{n-1}y^n)(1-x^{2n-1}y^{2n-3})(1-x^{2n-3}y^{2n-1})$$

$$(1.8)\qquad = \sum_{n=-\infty}^{+\infty}(-1)^n x^{(3n^2+n)/2}y^{(3n^2-5n)/2} + \sum_{-\infty}^{+\infty}(-1)^n x^{(3n^2-5n)/2}y^{(3n^2+n)/2}$$
$$+ \sum_{-\infty}^{+\infty}(-1)^{n+1}x^{(3n^2+n-2)/2}y^{(3n^2-5n+2)/2} + \sum_{-\infty}^{+\infty}(-1)^{n+1}x^{(3n^2-5n+2)/2}y^{(3n^2+n-2)/2}.$$

If $D(n, m)$ denotes the number of partitions of (n, m) into parts of the type $(2d-1, 2d-3)$, $(2e-3, 2e-1)$. We obtain a relation between $\alpha(n, m)$ and $D(n, m)$ by writing (1.8) in the form

$$\{\sum(-1)^\lambda(xy)^{\lambda(3\lambda+1)/2}\}\left\{\sum_{n,m=0}^{\infty}\alpha(n, m)x^ny^m\right\}$$
$$(1.9)\qquad = \left\{\sum_{n,m=0}^{\infty}D(n, m)x^ny^m\right\}\{\sum(-1)^n x^{(3n^2+n)/2}y^{(3n^2-5n)/2}$$
$$+ (-1)^n x^{(3n^2-5n)/2}y^{(3n^2+n)/2} + (-1)^{n+1}x^{(3n^2+n-2)/2}y^{(3n^2-5n+2)/2}$$
$$+ (-1)^{n+1}x^{(3n^2-5n+2)/2}y^{(3n^2+n-2)/2}\}$$

and equating coefficients.

1.5 can also be written as

$$\sum_{-\infty}^{+\infty}(-1)^n x^{n(n+1)/2}y^{n(n-1)/2} = \left\{\sum_{n,m=0}^{\infty}D(n, m)x^ny^m\right\}$$
$$(1.10)\qquad \cdot\left\{\sum_{-\infty}^{\infty}x^{(3n^2+n)/2}y^{(3n^2-5n)/2} + \sum_{-\infty}^{\infty}x^{(3n^2-5n)/2}y^{(3n^2+n)/2}\right.$$
$$\left. - \sum_{-\infty}^{+\infty}x^{(3n^2+n-2)/2}y^{(3n^2-5n+2)/2} - \sum x^{(3n^2-5n+2)/2}y^{(3n^2+n+2)/2}\right\},$$

equating coefficients

$$\sum D\left(n-\frac{3r^2+r}{2}, m-\frac{3r^2-5r}{2}\right)$$
$$+ \sum D\left(n-\frac{3r^2-5r}{2}, m-\frac{3r^2+r}{2}\right)$$
$$(1.11)\qquad - \sum D\left(n-\frac{3r^2-5r+2}{2}, m-\frac{3r^2+r-2}{2}\right)$$
$$- \sum D\left(n-\frac{3r^2+r-2}{2}, m-\frac{3r^2-5r+2}{2}\right) = (-1)^r \text{ or } 0$$

according as (n, m) is of the type $(r(r+1)/2, r(r-1)/2)$ or not. The Jacobi Identity

$$(1.12)\qquad \prod_{n=1}^{\infty}(1-x^ny^n)(1+x^{n-1}y^n)(1+x^ny^{n-1}) = \sum_{-\infty}^{+\infty}x^{n(n+1)/2}y^{n(n-1)/2}$$

can be written as

$$(1.13)\qquad \left\{\sum_{\lambda=0}^{\infty}(-1)^\lambda(xy)^{\lambda(3\lambda\pm1)/2}\right\}\left\{\sum_{n,m=0}^{\infty}\alpha(n, m)x^ny^m\right\} = \sum_{-\infty}^{+\infty}x^{r(r+1)/2}y^{r(r-1)/2}.$$

Thus equating coefficients

$$\sum_{\lambda} (-1)^{\lambda}\alpha \left(n - \lambda \frac{(3\lambda \pm 1)}{2}, m - \lambda \frac{(3\lambda \pm 1)}{2}\right) = 1 \text{ or } 0$$

according as (n, m) is or is not of the type $(r(r + 1)/2, r(r - 1)/2)$.

In the case of the number of partitions of an integer we have the well-known result that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts. We can prove the following generalization of this result for vector partitions.

THEOREM II. *The number of partitions of* $(n_1, n_2, \cdots, n_s)$ *into vectors with at least one component odd is equal to the number of partitions of* $(n_1, n_2, \cdots, n_s)$ *into distinct parts (vectors). Note, the same result holds if the parts are required to have non-zero components.*

Proof. Denote the generating function of unrestricted vector partitions by

$$\begin{aligned} f(x_1, x_2, \cdots, x_s) &= \prod_{k_i \geqq 0}^{\infty} (1 - x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})^{-1} \\ &= \sum u(n_1, n_2, \cdots, n_s) x_1^{n_1}, \cdots, x_s^{n_s} \end{aligned}$$

and notice that the generating function for the number of partitions with at least one component odd is

$$g(x_1, \cdots, x_s) \prod_{j_i \geqq 0} \{(1 - x_1^{j_1}, x_2^{j_2} \cdots x_s^{j_s})\}^{-1}$$

where at least one j_i is odd.

This is connected with $f(x_1, \cdots, x_s)$ by

$$g(x_1, \cdots, x_s) = \frac{f(x_1, \cdots, x_s)}{f(x_1^2, \cdots, x_s^2)} = \prod_{k_i \geqq 0} (1 + x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})$$

and this proves the result.

Let

$$f(x) = \left\{\prod_{n=1}^{\infty} (1 - x^n)\right\}^{-1},$$

$$g(x) = \frac{f(x)}{f(x^2)^2} = \sum_{n=0}^{\infty} x^{n(n+1)/2},$$

$$\theta(x) = \sum_{-\infty}^{+\infty} x^{n^2}.$$

Gordon [1] has shown that

$$F(x) = \frac{f(x^2)f(x^3)}{f(x)^2 f(x^6)} = g(x) - 3xg(x^9),$$

$$G(x) = \frac{f(x^2)f(x^3)f(x^{12})}{f(x)f(x^4)f(x^6)^2} = \frac{3}{2}\theta(x^9) - \tfrac{1}{2}\theta(x).$$

Thus

$$\int_0^1 x^{s^2-1} G(x)\,dx = \frac{3}{2}\sum_{-\infty}^{+\infty}\frac{1}{9n^2+s^2} - \frac{1}{2}\sum_{-\infty}^{+\infty}\frac{1}{n^2+s^2}$$

$$= \frac{\pi}{2s}\operatorname{Coth}\left(\frac{\pi s}{3}\right) - \frac{\pi}{2s}\operatorname{Coth}(\pi s)$$

when $s^2 \to 0$

$$\int_0^1 \{G(x) - 1\}\frac{dx}{x} = \frac{3}{2}\cdot 2\sum_{1}^{\infty}\frac{1}{9n^2} - \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= -\frac{2}{3}\cdot\frac{\pi^2}{6} = -\frac{\pi^2}{9}.$$

Also

$$\int_0^1 x^{s^2}\{F(x) - 1\}\frac{dx}{x} = \sum_{n=1}^{\infty}\frac{1}{\frac{n^2+n}{2}+s^2} - 3\sum_{n=1}^{\infty}\frac{1}{\frac{9n^2+9n+2}{2}+s^2}$$

when $s^2 \to 0$.

We obtain

$$\int_0^1 \{F(x) - 1\}\frac{dx}{x} = 2\sum_{n=1}^{\infty}\frac{1}{n^2+n} - 6\sum_{n=1}^{\infty}\frac{1}{9n^2+9n+2}$$

but $\sum_{n=1}^{\infty} 1/(n^2+n) = 1$. Thus

$$\int_0^1 \{F(x) - 1\}\frac{dx}{x} = 2 - 6\sum_{n=1}^{\infty}\frac{1}{9n^2+9n+2}.$$

The author recently extended the table of values $q_r(n, m)$ to $n, m = 1(1)\,49$, $r = 1(1)\,98$. These tables display the unique maxima property of $P_r(n, m)$ the number of partitions of (n, m) into exactly r parts with positive components. Szekeres [3] proved this result for $P_r(n)$ the number of partitions of n into exactly r parts. The value of $r = r_0$ for which such a maxima occurs was also obtained by Szekeres. It seems reasonable to conjecture that $P_r(n_1, n_2, \cdots, n_s)$ attains a unique maxima for fixed n_i and r varying, to locate the position of the maxima is still another problem, we hope these numerical results will be useful in establishing these results. Here we list the values of $q_r(n, m)$ and $P_r(n, m)$. The number of partitions of (n, m) into a most r parts and into exactly r parts with positive components respectively for $n = m = 49$, $r = 1(1)\,98$. These calculations were performed on IBM7072 at the University of Arizona Computing Centre.

r	q_r (49, 49)	r	P_r(49, 49)
1	1	1	1
2	1250	2	1152
3	2 71250	3	2 12352
4	204 56138	4	125 40912
5	7253 82374	5	3213 83504
6	1 43784 40981	6	42554 50133
7	17 84775 75068	7	3 27015 33936

r	$q_r(49, 49)$	r	$P_r(49, 49)$
8	150 48471 17166	8	15 87937 47277
9	916 77113 62366	9	52 08652 47085
10	4234 98219 30204	10	121 95462 10853
11	15473 71039 36275	11	213 70941 87417
12	46147 39455 92382	12	292 23660 88026
13	1 15673 58257 26397	13	323 71338 03249
14	2 49776 71698 61341	14	300 40681 24568
15	4 74660 54906 63150	15	240 71325 68777
16	8 08754 69334 51823	16	171 08154 40544
17	12 55778 74241 16759	17	110 42127 59320
18	18 02318 70750 55031	18	66 04574 73208
19	24 20668 26439 28450	19	37 23585 89548
20	30 75237 65765 05129	20	20 06417 05470
21	37 29788 68910 41293	21	10 44661 76145
22	43 53218 32697 73604	22	5 29944 33056
23	49 22819 19823 33522	23	2 63515 05294
24	54 25075 38047 75435	24	1 28971 50552
25	58 54695 63056 88644	25	62293 44602
26	62 12736 16879 48138	26	29736 41660
27	65 04509 68770 92683	27	14038 75228
28	67 37716 16793 14191	28	6555 17660
29	69 20992 31811 86572	29	3026 58703
30	70 62911 29602 74915	30	1380 94202
31	71 71374 52344 73808	31	622 29879
32	72 53303 45626 50671	32	276 69589
33	73 14538 55406 77176	33	121 30780
34	73 59867 76299 09253	34	52 36586
35	73 93126 40369 25937	35	22 24235
36	74 17328 64517 38952	36	9 27622
37	74 34805 42592 48296	37	3 79693
38	74 47334 40431 53499	38	1 51958
39	74 56254 78400 31912	39	59521
40	74 62564 46471 11855	40	22652
41	74 66999 66842 87939	41	8406
42	74 70098 61871 08998	42	2998
43	74 72251 46591 09466	43	1043
44	74 73738 78617 19263	44	339
45	74 74760 84919 25277	45	109
46	74 75459 59174 26951	46	31
47	74 75934 93051 42650	47	9
48	74 76256 74971 94430	48	2
49	74 76473 62747 36410	49	1

r	$q_r(49, 49)$	r	$q_r(49, 49)$
50	74 76619 13284 07810	61	74 76902 76609 81020
51	74 76716 33243 94039	62	74 76903 64644 21013
52	74 76780 99081 70684	63	74 76904 20650 16696
53	74 76823 82642 71331	64	74 76904 56120 10320
54	74 76852 09087 92024	65	74 76904 78479 57187
55	74 76870 66707 26980	66	74 76904 92506 52774
56	74 76882 82803 01034	67	74 76905 01262 04927
57	74 76890 75809 13813	68	74 76905 06698 67949
58	74 76895 90889 79998	69	74 76905 10056 10432
59	74 76899 24120 51895	70	74 76905 12117 72942
60	74 76901 38833 23182	71	74 76905 13376 13392

r	$q_r(49, 49)$
72	74 76905 14139 47711
73	74 76905 14599 48401
74	74 76905 14874 79083
75	74 76905 15038 35975
76	74 76905 15134 79989
77	74 76905 15191 19820
78	74 76905 15223 89783
79	74 76905 15242 68239
80	74 76905 15253 36887
81	74 76905 15259 38411
82	74 76905 15262 73249
83	74 76905 15264 57338
84	74 76905 15265 57247
85	74 76905 15266 10672
86	74 76905 15266 38810
87	74 76905 15266 53360
88	74 76905 15266 60749
89	74 76905 15266 64414
90	74 76905 15266 66192
91	74 76905 15266 67072
92	74 76905 15266 67409
93	74 76905 15266 67575
94	74 76905 15266 67645
95	74 76905 15266 67672
96	74 76905 15266 67682
97	74 76905 15266 67685
98	74 76905 15266 67686

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